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Course Notes

M332 COURSE NOTES 1

(1) Broadcasts

Television programme 3 covers various topics concerned with complex integration. Television programme 4 discusses the results of Unit 6, in particular, Taylor's Theorem. Radio programme 3 concentrates on techniques for calculating Taylor series. Refer to the Supplementary Material for further details.

(2) Specimen Question

You may treat the following question as a non-assessed assignment question if you wish. The solution, comments and mark scheme are supplied for self assessment.

1. (i) Prove that the set $\left\{\left|\frac{1}{4z^2 + 3}\right| : |z| = 2\right\}$ has least upper bound $\frac{1}{13}$. [5]

(ii) Let C be the circle $\{z : |z| = 2\}$ and let I be the integral

$$I = \int_C \frac{z + 5}{4z^2 + 3}\, dz.$$

Use the Estimation Theorem (Theorem 10, page 47) to obtain an upper bound for $|I|$. (Show all your working.) [3]

Solution and mark-scheme

The boxed-in comments are for your understanding and do not form part of the solution.

(i)

We find an upper bound for $\left\|\frac{1}{4z^2 + 3}\right\|$ by finding a lower bound for $\|4z^2 + 3\|$.	
For all complex numbers α, β $\|\alpha - \beta\| \geq \|\|\alpha\| - \|\beta\|\|$. Now $\|4z^2 + 3\| = \|4z^2 - (-3)\|$ $\geq \|\|4z^2\| - \|-3\|\|$ $= \|4\|z\|^2 - 3\|$.	1 mark for obvious use of this equality. 1 mark for this far.

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Hence, for $|z| = 2$,

$$|4z^2 + 3| \geq |16 - 3| = 13;$$

and so

$$\left|\frac{1}{4z^2 + 3}\right| \leq \frac{1}{13}, \quad |z| = 2.$$

1 mark for getting upper bound.

Note that so far we have only found an upper bound. We are required to prove it is the least upper bound.

For $z = 2i$,

$$|4z^2 + 3| = |-16 + 3| = 13,$$

and so

$$\left|\frac{1}{4z^2 + 3}\right| = \frac{1}{13}$$

½ mark for choice of z with $|z| = 2$.

½ mark for working through.

Hence the upper bound is attained and is therefore the least upper bound.

1 mark for justification.

(ii) By the Estimation Theorem,

$$|I| \leq ML$$

where M is an upper bound of $|f|$ on C and L is the length of the contour.

$L = 2\pi(2) = 4\pi$.

½ mark.

On C,

$$\left|\frac{z + 5}{4z^2 + 3}\right| \leq \frac{|z| + 5}{|4z^2 + 3|}$$

$$\leq \frac{2 + 5}{13}, \text{ by the result of part (i).}$$

1 mark for correct use of theorem.

Hence,

$$|I| = \left|\int_C \frac{z + 5}{4z^2 + 3}\, dz\right|$$

$$\leq \frac{7}{13} \cdot 4\pi$$

$$= \frac{28\pi}{13}.$$

1 mark for any correct estimate.

½ for accuracy of result.

Note that any correct estimate of f on C will do, but as we have already caculated the least upper bound of $\left|\frac{1}{4z^2 + 3}\right|$ on C, we may as well use it.

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Course Notes

M332 COURSE NOTES 2

(1) Specimen Questions

As in previous Stop Presses we include the following questions which you may like to use as non-assessed assignment questions. The solutions, comments and mark-schemes are supplied for self-assessment.

1. (i) Define the term 'Cauchy region'. [1]

   (ii) Let $S = \{z: |z - a| + |z - b| \leq k,\ a,b \in \mathbb{C},\ k > |a - b|\}$ and let $\alpha \in S$. Show that there exists a circle centre α containing S and, using only the definition of part (i), prove that $R = \mathbb{C} - S$ is not a Cauchy region. [5]

2. Find the maximum value of $z \to |\exp(z^4)|$ on the disc $|z| \leq 3$ and determine all those points at which this maximum is achieved. [5]

Solutions and mark-scheme

1. (i) A region R is said to be a Cauchy region if whenever Γ is a closed contour in R and f is analytic on R then

$$\int_\Gamma f = 0.$$ [1 mark]

(ii)

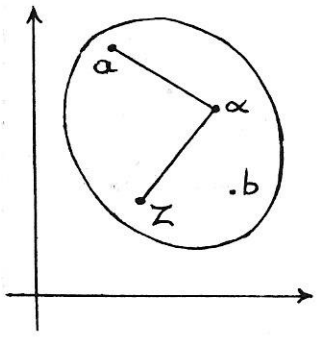
> Comments: We are required to find a value of ρ such that $S \subseteq \{z: |z - a| \leq \rho\}$.

For all $z \in S$,

$$|z - a| + |z - b| \leq k,$$

and so

$$|z - a| \leq k \text{ and } |z - b| \leq k.$$



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Now for all $z \in S$

$$|z - \alpha| = |z - \alpha + a - a|$$
$$\leq |z - a| + |\alpha - a|$$
$$\leq k + |\alpha - a|.$$

Similarly, $|z - \alpha| \leq k + |\alpha - b|$ and so,

if $\rho > \max(k + |\alpha - a|, k + |\alpha - b|)$,

then $|z - \alpha| < \rho$

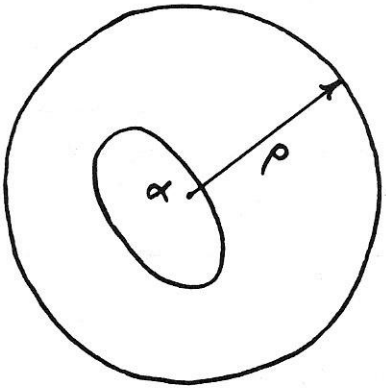
for all $z \in S$.

Hence the circle $C = \{z: |z - \alpha| = \rho\}$ is the required circle.

½ mark for use of inequality (or correct argument using diagram).

1 mark for any correct value of ρ.

Comment: The trick $|z - \alpha| = |z - \alpha + a - a| \leq |z - a| + |\alpha - a|$ is a useful technique to remember. Since we are allowed to use only the definition of Cauchy region we cannot appeal to Theorems 5 and 6 of Unit 9 to say that Cauchy regions and simply-connected regions are equivalent (for then of course the proof would be trivial since R is not simply-connected). Since we are required to prove that R is not a Cauchy region we must find a closed contour Γ and a function f analytic on R such that $\int_\Gamma f \neq 0$; in other words, our proof will be by counter-example. Moreover, the circle C looks as though it will be a reasonable choice for Γ.



Let f be the function defined by

$$f(z) = \frac{1}{z - \alpha}, \quad z \neq \alpha.$$

Then f is analytic on R and

$$\int_C \frac{1}{z - \alpha}\, dz = \int_0^{2\pi} \frac{1}{\rho e^{i\theta}}\, i\rho e^{i\theta} d\theta$$
$$= 2\pi i.$$

Hence f is a function on R for which $\int_C f \neq 0$ and so R is not a Cauchy region.

1 mark for choosing a suitable function.

½ for checking condition.

1 mark for correct evaluation of the integral.

1 mark for the conclusion.

Total 5

Comment: Any function with the required property could be chosen, but it is as well to make it as simple as possible. We are led to the above choice by a number of pointers.

(i) Winding number. We know that $\mathrm{Wnd}(\Gamma,\alpha) = \int_\Gamma \frac{dz}{z-\alpha}$ (see page 84 of Unit 9) and we can see from the diagram that $\mathrm{Wnd}(C,\alpha) = 1$.

(ii) As we are looking for a non-zero integral we know from the Residue Theorem (page 102 of Unit 9) that we require a function having a non-zero residue at α. What could be simpler than $1/(z-\alpha)$?

2. Since $z \to \exp(z^4)$ is analytic (in fact, entire) and continuous on $|z| \leq 3$, we may apply the Maximum Principle (page 62 of Unit 9) to conclude that the maximum of $|\exp(z^4)|$ occurs on the boundary of the disc, that is, on $|z| = 3$.

½ mark for checking condition.

1 mark for applying Maximum Principle correctly.

Note: It is necessary to check the conditions of a theorem are satisfied before invoking it, even in such an obvious case as this!

Put $z = 3e^{i\theta}$. Then,

$$\begin{aligned} |\exp(z^4)| &= |\exp(81e^{4i\theta})| \\ &= |\exp(81\cos 4\theta + 81i\sin 4\theta)| \\ &= |\exp(81\cos 4\theta)|.|\exp(81i\sin 4\theta)| \\ &= e^{81\cos 4\theta}, \text{ since } |\exp(81i\sin 4\theta)| = 1. \end{aligned}$$

1 mark for correct method.

½ mark for accurate working.

Now the maximum value of $e^{81\cos 4\theta}$ occurs when

$$\cos 4\theta = 1;$$

½ mark for this.

that is, when

$$\theta = \frac{k\pi}{2},\ k = 0, \pm 1, \pm 2, \ldots,$$

½ mark for correct values of θ.

and has the value e^{81}. The points at which this occurs are $z = 3e^{i\theta}$, $\theta = \frac{k\pi}{2}$, $k = 0, \pm 1, \pm 2, \ldots$, that is $z = 3$, -3, $3i$ and $-3i$.

½ mark for maximum value.

½ mark for correct points.

Total 5

Note: The maximum occurs at more than one point and all these points must be given.

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Course Notes

M332 COURSE NOTES 3

(1) Specimen Question

The following question is a theory-type question which tests (amongst other things) Rouché's Theorem. When attempting the question, take care over your choice for the functions 'f' and 'g' of Rouché's Theorem. If you cannot do the problem, try to draw up a plan of campaign and compare it with the comments given in the mark scheme.

Question

Let $\{h_n\}$ be a sequence of functions each analytic on a region R, and such that $\{h_n\}$ converges uniformly to h on every closed disc in R.

(i) Suppose that z_0 is a zero of h and that h is not identically zero on R. Show that there exists a disc $\{z: |z - z_0| < \delta\}$ which contains no other zeros of h.

(ii) Suppose now that h has at least one zero in R. Use the result of (i), together with Rouché's Theorem, to show that, if each h_n has no zeros in R, then h is identically zero on R. [10]

Solution and mark scheme

(i) Since $\{h_n\}$ is a sequence of analytic functions uniformly convergent to h on every closed disc in R, we have, by Theorem 3 of Unit 11, that h is analytic on R. Since $h \neq 0$ on R, it follows from Corollary 1, Theorem 15 of Unit 6 that any zero of h is isolated: in other words, there is a disc $\{z: |z - z_0| < \delta\}$ containing no other zeros of h.

1M for h analytic

1M for isolated zeros

1M for conclusion

If you have difficulty in part (ii) read the following comment.

(ii)

Comment

In view of part (i) it looks as if a reasonable approach would be a proof by contradiction, i.e. we shall assume that h is non-zero. In essence, (i) says that if h is not identically zero, then given a zero of h we can find a disc which contains only this zero. But within this disc h_n has no zeros. If we could now apply Rouché's Theorem with f and f + g being the functions h and h_n we would have the required contradiction. Our only difficulty seems to be whether to choose h or h_n as our 'f', and consequently, $h_n - h$ or $h - h_n$ as our 'g'. In order to apply Rouché's Theorem we need to show '$|f| > |g|$' on some simple-closed contour. The contour we choose must lie in the disc obtained in part (i), and this suggests

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that we choose h to be the function f. With this choice, $g = h_n - h$ and we know that we can make $|h_n(z) - h(z)|$ as small we we please. If we can get $|h(z)| > k$, where $k > 0$, on our chosen contour, then the inequality will be proved. The latter looks promising if we recall that a function continuous on a compact set (the chosen contour) is bounded on that set. We use this fact first.

Suppose h is not identically zero and that z_0 is a zero of h. Then by (i) there exists a disc $D = \{z: |z - z_0| < \delta\}$ containing no other roots of f. Let $C = \{z: |z - z_0| = \delta_1, 0 < \delta_1 < \delta\}$.

1M for method of contradiction

We choose C inside D to be the simple-closed contour required for Rouché's Theorem.

Now h is non-zero and continuous on C, and so 1/h is continuous on C. But C is compact, and so, by the Boundedness Theorem (Theorem 12 of Unit 2), 1/h is bounded on C. In other words there eixsts $k \in \mathbb{R}$ such that

$$\left|\frac{1}{h(z)}\right| < k \quad (z \in C)$$

i.e. $|h(z)| > 1/k = \rho$ say.

$\frac{1}{2}$M

1M for theorem

$\frac{1}{2}$M for result

Note that we require C to be compact here. The result does not necessarily follow if C is not compact, as the following example shows. Take $S = (0, 1)$ and $f: x \to x$ $(x \in \mathbb{R})$. Then S is not compact, and $f(x) > 0$ for all $x \in S$. However it is not true that $f(x) > k$, $x \in S$, for some $k > 0$. We now show that $|h_n(z) - h(z)| < \rho$ for sufficiently large n, whence we will have shown that '$|f| > |g|$' on C.

Since $\{h_n\}$ converges uniformly to h on C, there exists an N such that, for all $z \in C$ and $n > N$, we have

$$|h_n(z) - h(z)| < \rho.$$

Hence, for $z \in C$ and $n > N$,

$$|h(z)| > |h_n(z) - h(z)|.$$

1M for using uniform convergence

1M for result

Thus we have proved '$|f| > |g|$'. Since $f = h$ and $g = h_n - h$ are analytic on D, we can conclude that $f = h$ and $f + g = h_n$ have the same number of zeros inside C.

Now h and $h_n - h$ are analytic on a simply-connected region containing C, and so by Rouché's Theorem h and $h + (h_n - h)$ have the same number of zeros inside C. But $h + (h_n - h) = h_n$, so h_n and h have the same number of zeros inside C But this contradicts the hypothesis that h_n has no zeros in R and so our assumption was incorrect. Hence h is identically zero.

1M for correct use of Rouche's Theorem

1M for accurate argument

Total 10

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Course Notes

M332 COURSE NOTES 4

M332 CN4

(1) Specimen Question

We have not included specimen questions here. If you would like to see worked examples on Units 13 and 14 please refer to the Specimen Examination Paper.

(2) Specimen Examination Paper

We enclose a Specimen Examination Paper in this mailing, together with solutions and a mark-scheme, for you to use for practice and revision. Part I carries 50 marks, Part II carries 34 marks and Part III carries 16 marks.

We have tried to make the task of memorizing theorems and definitions easier, by giving you advanced notice (through the bookmarks) of those results which we expect you to know.

(3) Finally, we hope you have enjoyed this course. If you have any suggestions we are always pleased to hear from you. Good luck in the exam.

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